Intoduction to topological order and topologial quantum computation

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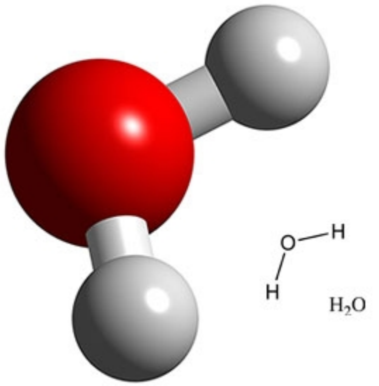
Outline

1. Introduction: phase transitions and order.
2. The Landau symmetry breaking description.
3. The topological order.
4. Signs of topological order: anyons.
5. Topological quantum computation.

Introduction: Phase transitions and the concept of order

The principle of emergence

Matter is formed by atoms/molecules.

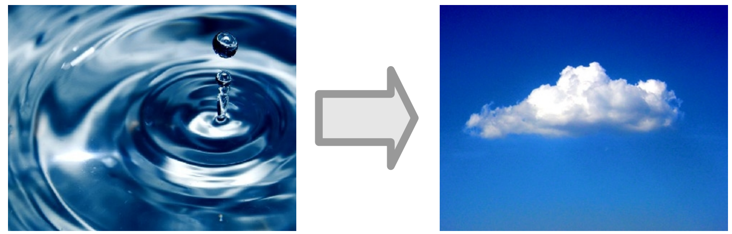


Different organizations ⇨ Different collective properties

Each of these organizations is called phase.

Phase Transitions

A phase transition is the transformation of a thermodynamic system from one phase to another.



In a phase transition, the free energy or some of its derivatives diverge.

First order PTs
involve a latent heat

Second order PTs
do not involve a latent heat

Quantum phase transitions

A quantum phase transition is a phase transition between two quantum phases at zero temperature. They can only be accessed by varying a physical parameter.

We will identify any point of nonanalyticity in the ground state energy of the infinite system as a quantum phase transition.

Example: The quantum Ising model. $$H_{\mathrm{Ising}} = -\frac{1}{2}\sum_l \left(\sigma_l^x \sigma_{l+1}^x + \lambda \sigma_l^z\right)$$

Quantum paramagnet phase

$$|GS\rangle = \prod_i |\uparrow\rangle_i$$

$$\langle GS|\sigma_i^x \sigma_j^x|GS\rangle \sim e^{-|x_i - x_j|/\xi}$$

Quantum ferromagnet phase

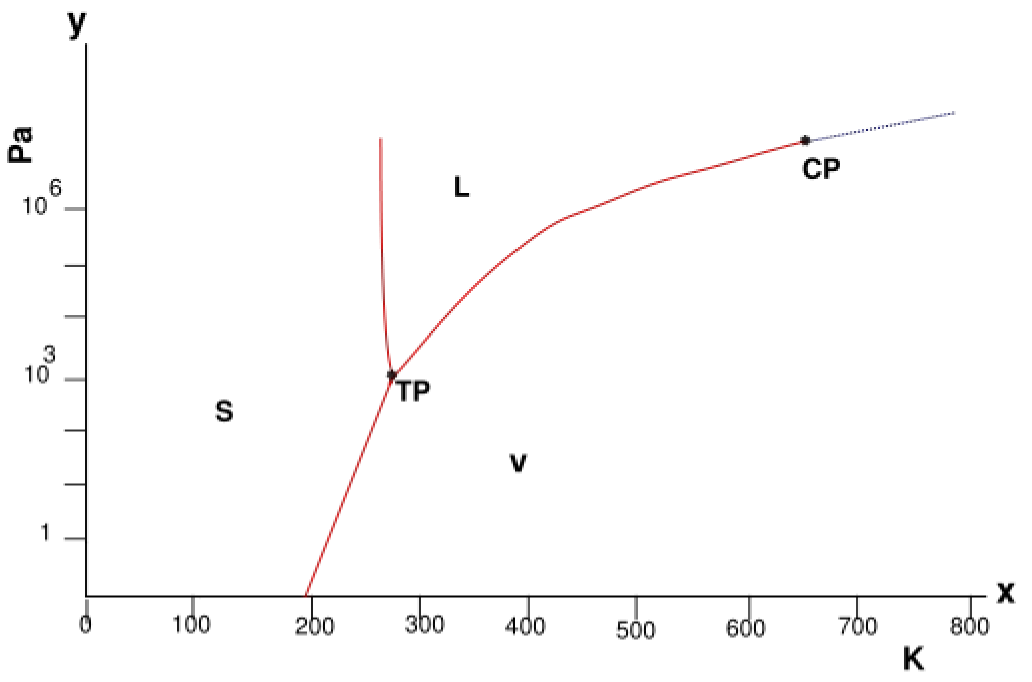
$$|GS\rangle = \prod_i |\leftarrow\rangle_i$$

$$\lim_{|x_i - x_j| \to \infty} \langle GS|\sigma_i^x \sigma_j^x|GS\rangle \sim N_0^2$$

The concept of order

The concept of order involves phase transition.

Two many particle states are in the same order if we can smoothly change one state into the other without encountering a phase transition.



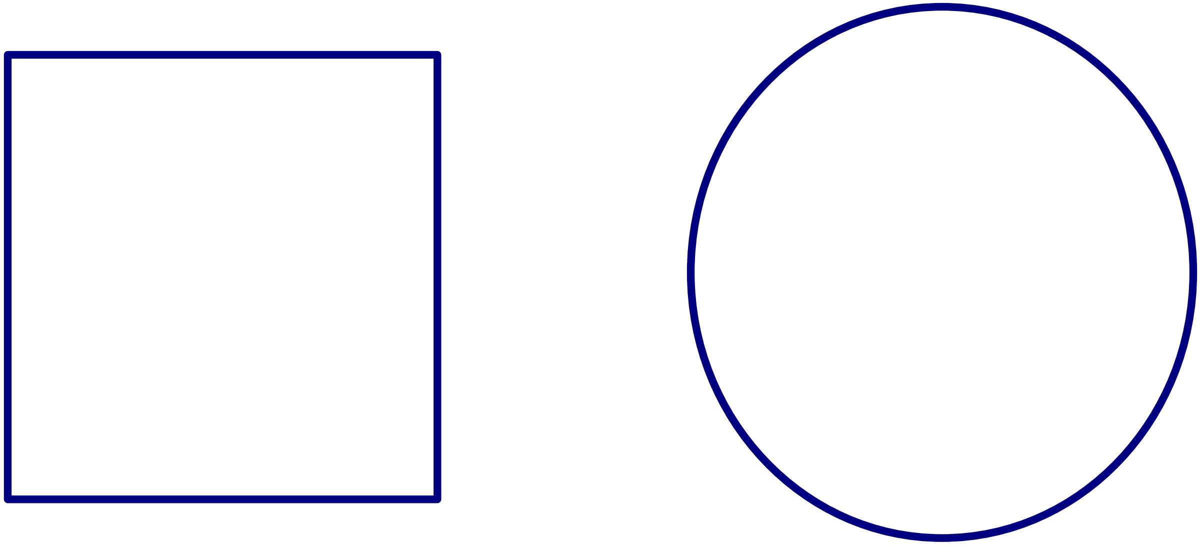
How many orders we have in the water phase diagram?

What makes two orders really different?

Landau symmetry breaking theory

Order and symmetry

Order and symmetry are complementary concepts.



"More order means less symmetry"

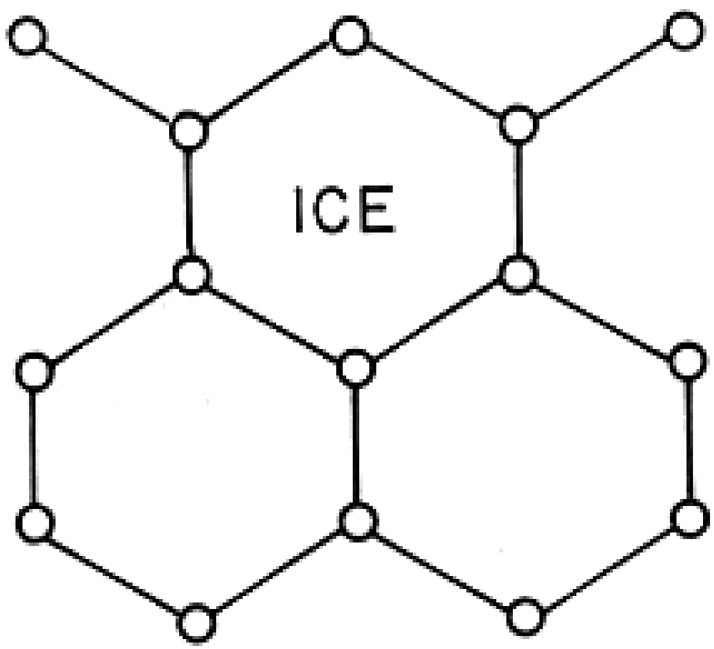
Order and symmetry

Order and symmetry are complementary concepts.

"More order means less symmetry"

Order and symmetry

Which phase is more symmetric?



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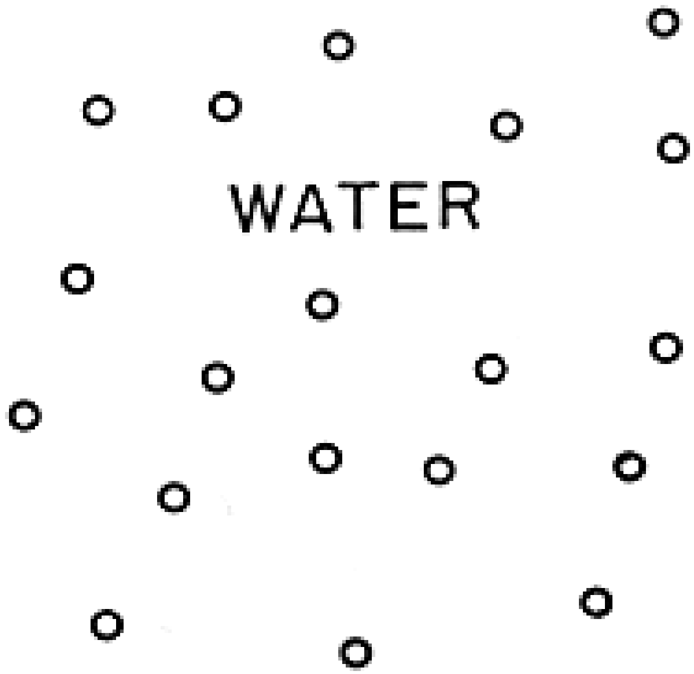

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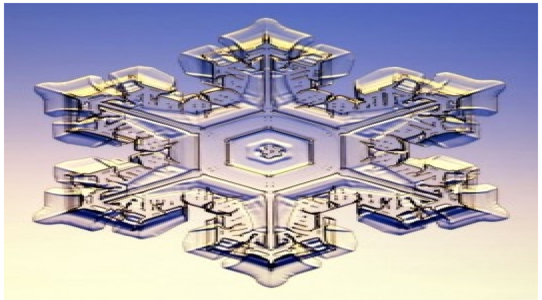
Landau symmetry breaking theory

The Landau symmetry breaking theory relates order with symmetry.

Different orders correspond to different symmetries



Discrete translational invariance

Continious translational invariance

Symmetry breaking

Phonons are the Goldstone bosons associated to the symmetry breaking.

Order parameter

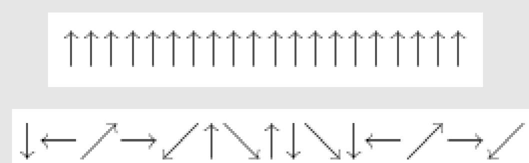
The order parameter is a quantity that:

- takes one value in one phase and another one in the other to distinguish in which phase the system is.
- measures of the degree of order in a system; the extreme values are 0 for total disorder and 1 for complete order.

Example

In the Heisenberg model, the magnetization is 0 in the paramagnetic phase and 1 in the ferromagnetic one.

$$H = -\sum_i \vec{S}_i \vec{S}_{i+1}$$



Which is the broken symmetry?

Landau's paradigm

The Landau's paradigm ingredients:

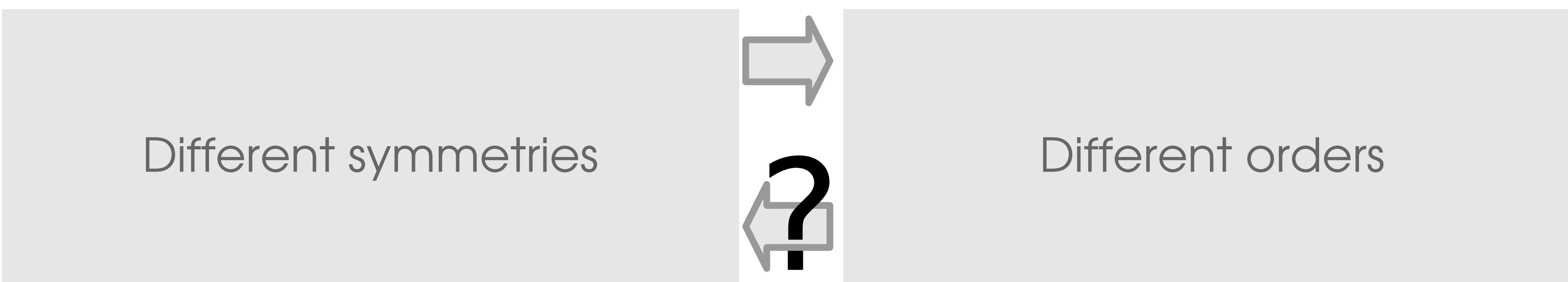
- the relation between symmetry (group theory) and order.
- the origin of many gapless excitations: phonons, spin waves,...
- the order parameter.

The Landau theory has been very successful:

- we can classify all of the 230 different kinds of cristals that exists in 3 dimensions.
- by determining how symmetry changes across a 2nd order phase transition, we can obtain its critical properties.

Beyond Landau's paradigm: topological order

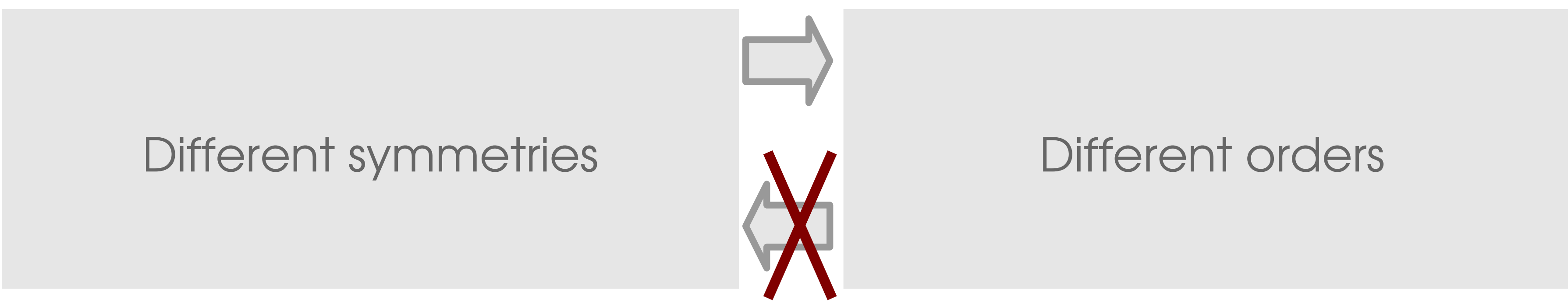
Can two different orders have the same symmetry?



Beyond Landau's paradigm

Can two different orders have the same symmetry?

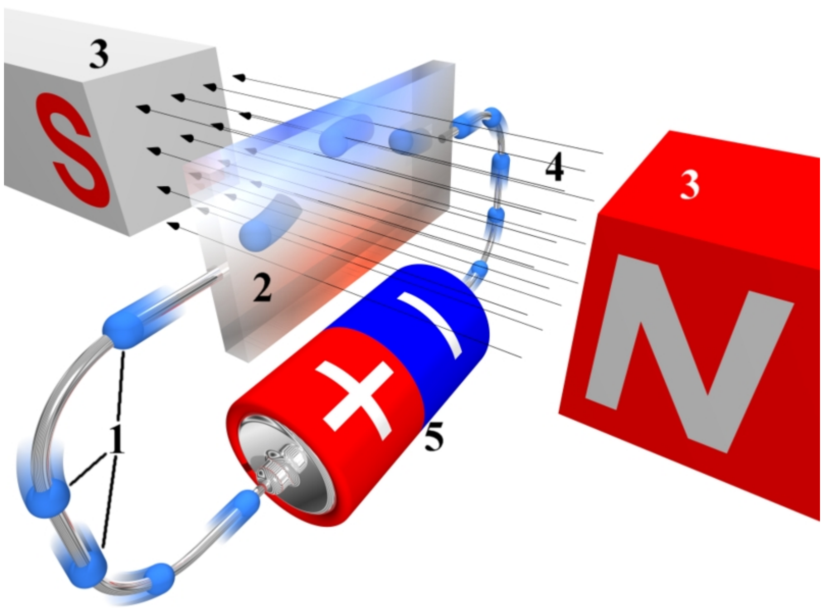
yes!

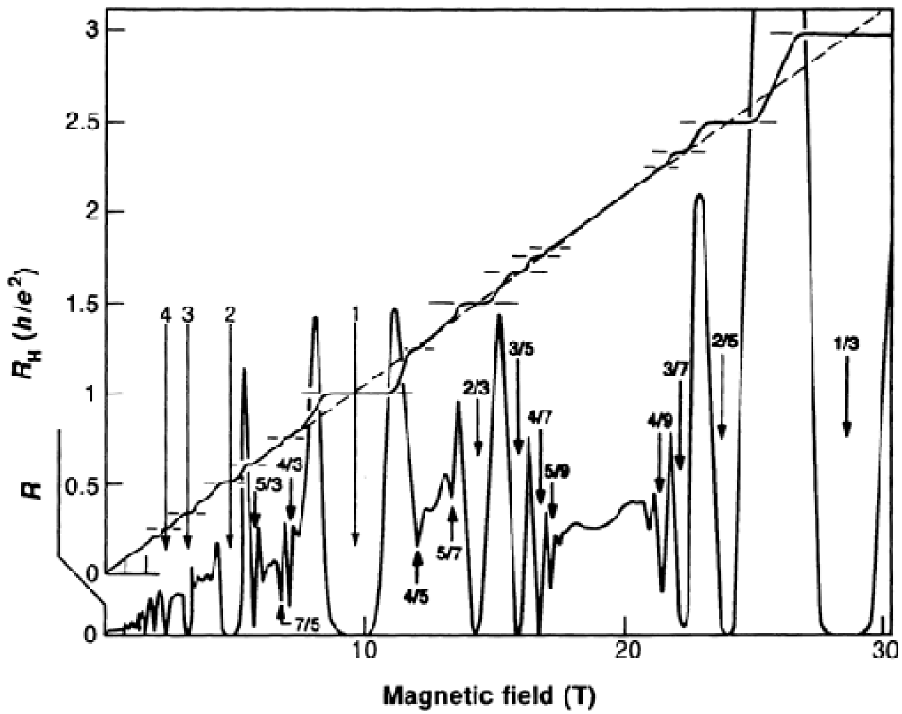


The Fractional Quantum Hall Effect (FQHE)!

The Fractional Quantum Hall Effect

In 1980, Tsui *et al.* measured the transversed resistivity of a 2D electron gas (at interfaces between semiconductors) under a strong transverse magnetic field at very low temperatures.





They observed the formation of *plateaus*, where the Hall conductivity took values 1/3, 5/2, 2/3 of e^2/h .

The Laughlin wave function

The Laughlin wave function was postulated by Robert B. Laughlin in 1983 as the ground state of the FQHE. It is defined by

$$\Psi_L^{(m)}(z_1, \ldots, z_n) \sim \prod_{i<j} (z_i - z_j)^m e^{-\sum_{i=1}^N |z_i|^2/2}$$

where $z_i = x_i + i\, y_i$.

It takes into account:

- the single-electron Hilbert space is restricted to the subspace of zero energy (lowest Landau level).
- The statistics of the particles.
- The repulsive interaction between electrons.

The Laughlin wave function

Why the Laughlin state is so special?

$$\Psi_L^{(m)}(z_1, \ldots, z_n) \sim \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^N |z_i|^2/2}$$

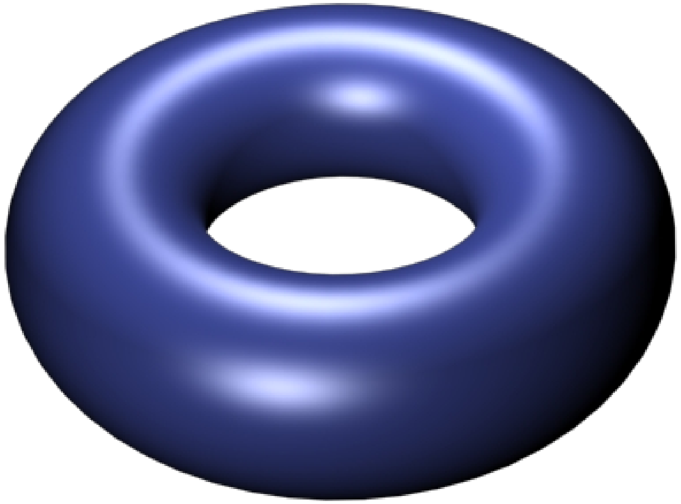
This state is is completely disorderd and symmetry is not broken.

States with a different m share the same symmetry and we cannot change one into another without finding a quantum phase transition.

The topological order

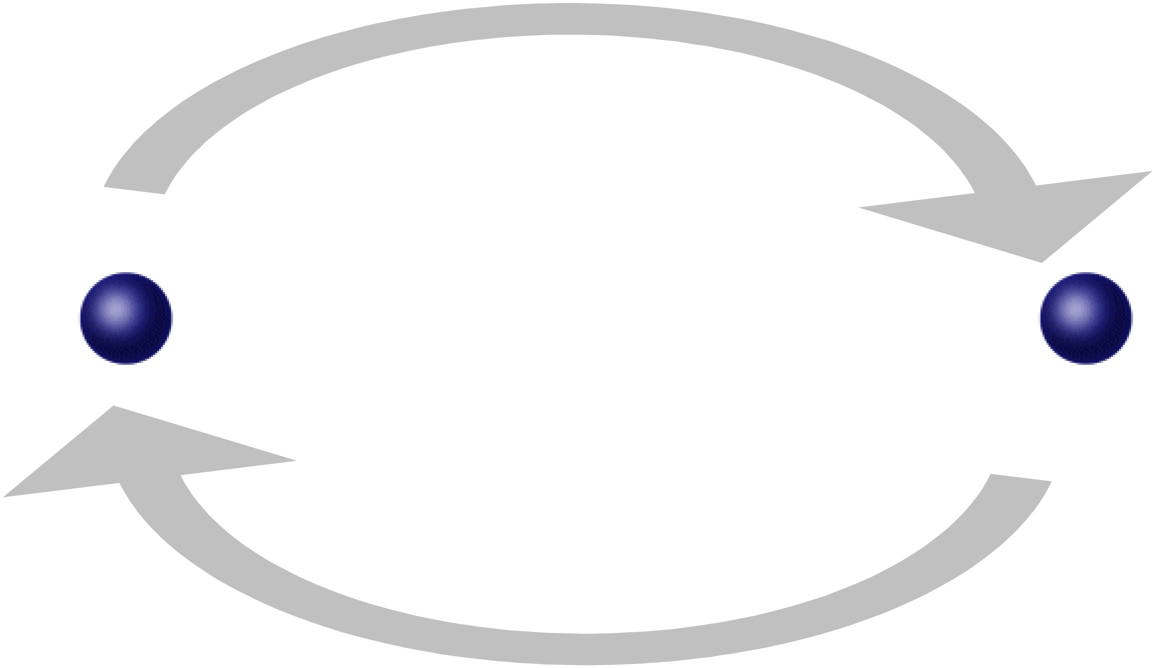
The topological order (TO) is defined by 3 features:

- the ground state degeneracy depends on the topology of space.
- there are anyonic quasiparticle excitations.
- there are edge excitations – a practical way to measure TO.



Anyons

Fermions and bosons



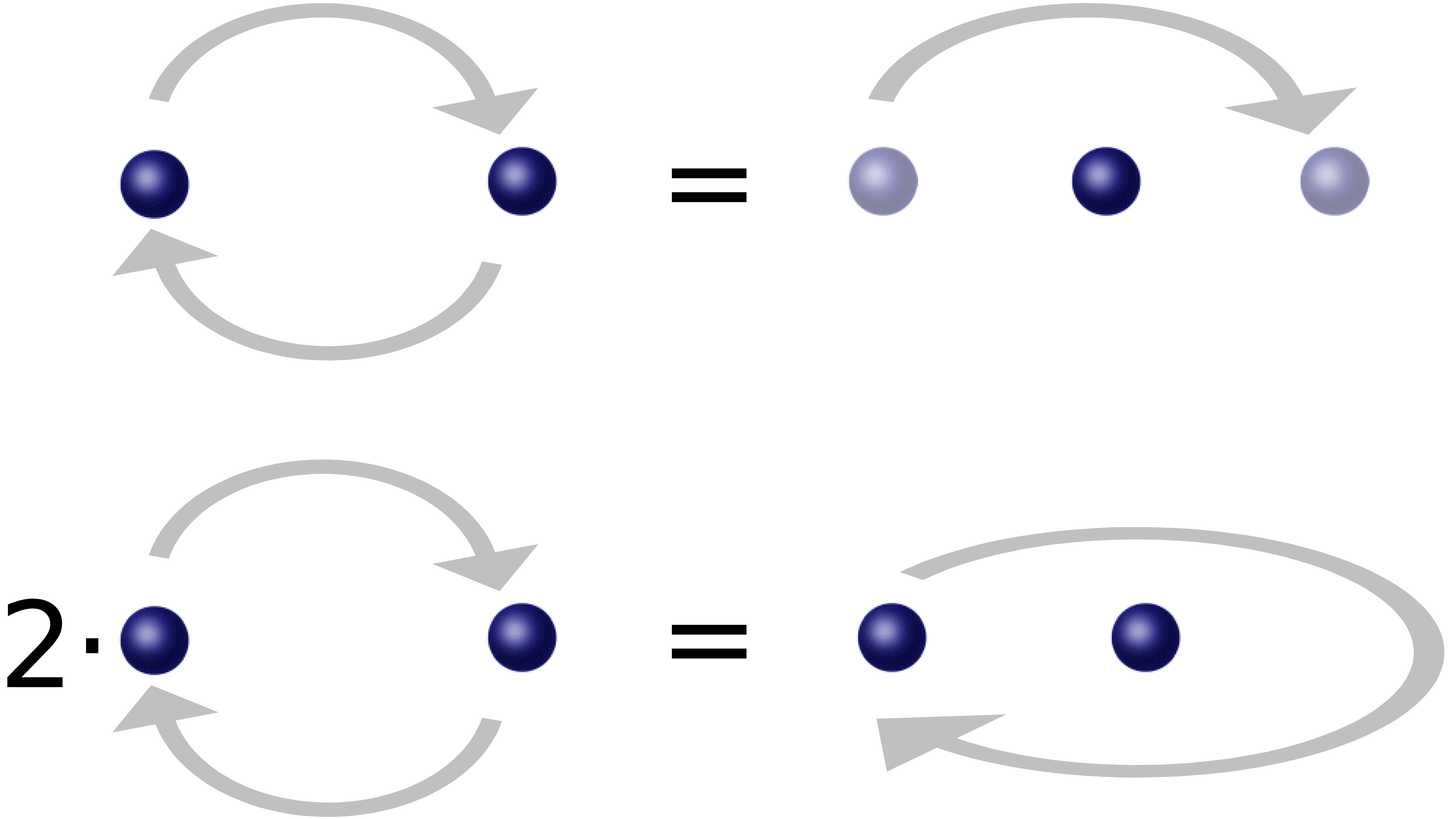
Bosons

$$|b_1\, b_2\rangle = |b_2\, b_1\rangle$$

Fermions

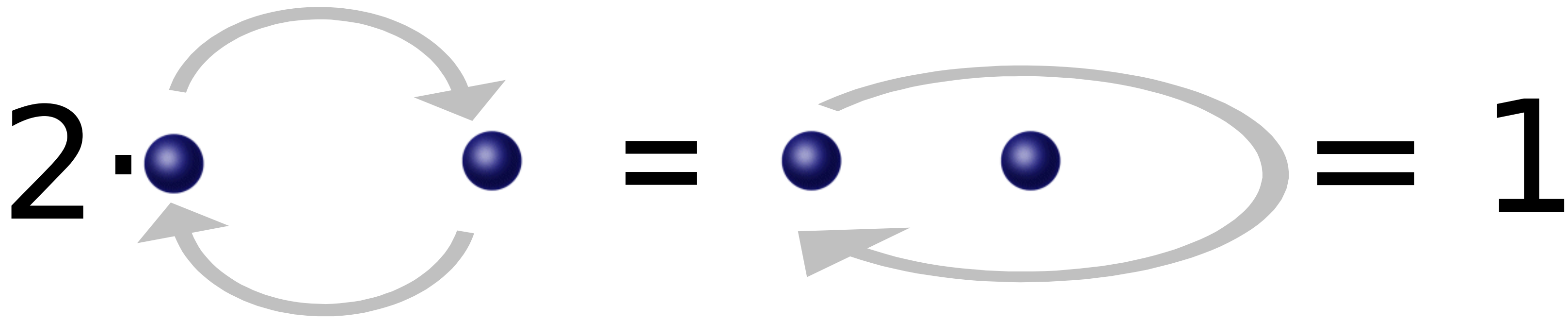
$$|f_1\, f_2\rangle = -\,|f_2\, f_1\rangle$$

Loops and interchanges

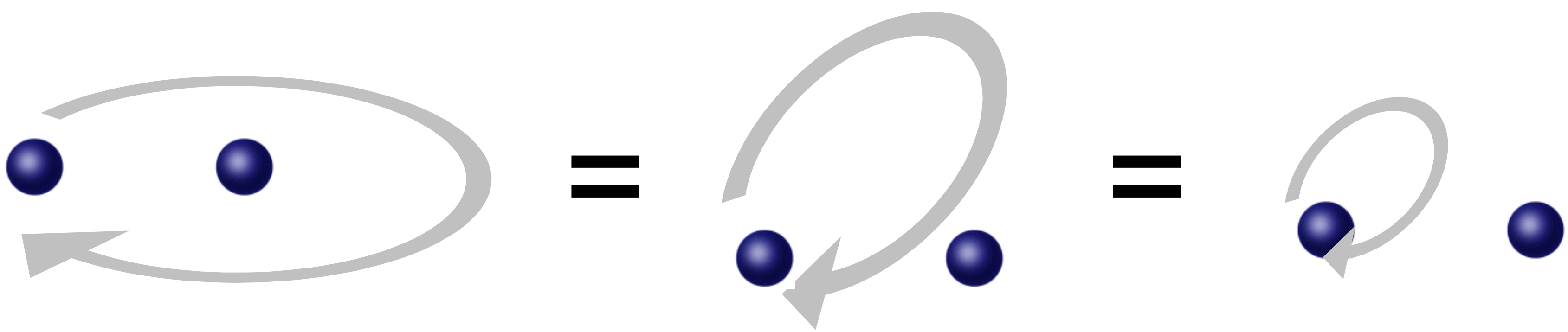


Loops and interchanges

In 3 dimensions:



since we can continuously deform the loop arround the other particle into the identity.

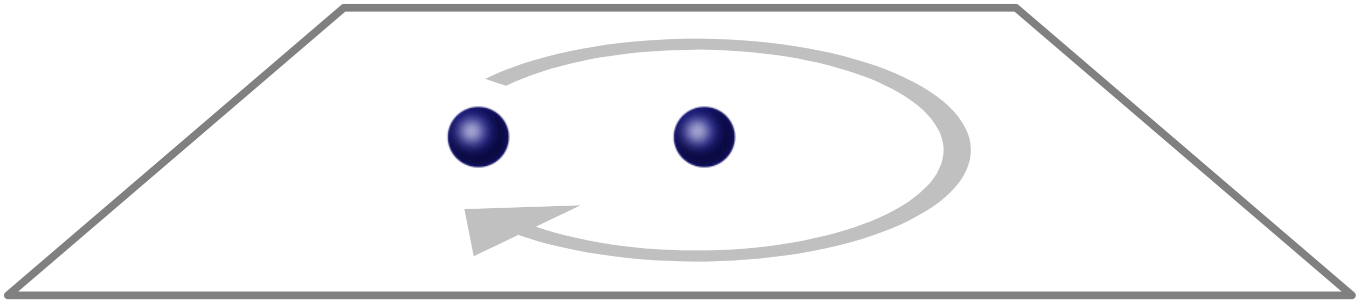


This is the reason why in 3 dimensions we can only have bosons or fermions.

$$|p_1\, p_2\rangle = e^{i\theta}\, |p_2\, p_1\rangle \;\Rightarrow\; e^{i2\theta}\, |p_1\, p_2\rangle = |p_1\, p_2\rangle \;\Rightarrow\; e^{i\theta} = \pm 1$$

Anyons

Nevertheless, in 2 dimensions we cannot contract the loop into the identity.



This makes possible the existence of particles that are neither bosons nor fermions. We will call them anyons.

$$|a_1\, a_2\rangle = e^{i\theta}\, |a_2\, a_1\rangle$$

Abelian and non-abelian anyons

Let us assume that our particles have an internal degree of freedom (colour, spin, etc.).

Then, the most general operation that the particles would suffer under an interchange would be,

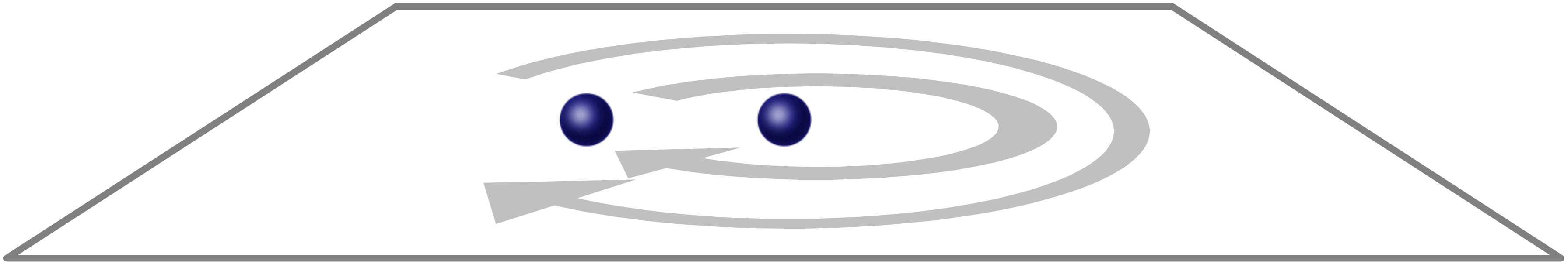
$$|a_1^{\alpha}\, a_2^{\beta}\rangle = U_{\gamma\delta}^{\alpha\beta}\, |a_2^{\gamma}\, a_1^{\delta}\rangle$$

where $U_{\gamma\delta}^{\alpha\beta}$ is a unitary matrix.

Abelian anyons	Non-abelian anyons
The generators of *U* belong to an abelian group	The generators of *U* do not belong to an abelian group

Applications of anyons

We have a robust method of performing unitary operations on quantum states.



$$|a_1^\alpha \; a_2^\beta\rangle = U^{\alpha\beta}_{\gamma\delta} \, |a_2^\gamma \; a_1^\delta\rangle$$

The final unitary operation performed does not depend on the particular path followed arround the particle.

Can we take profit of this technology?

Topological quantum computation

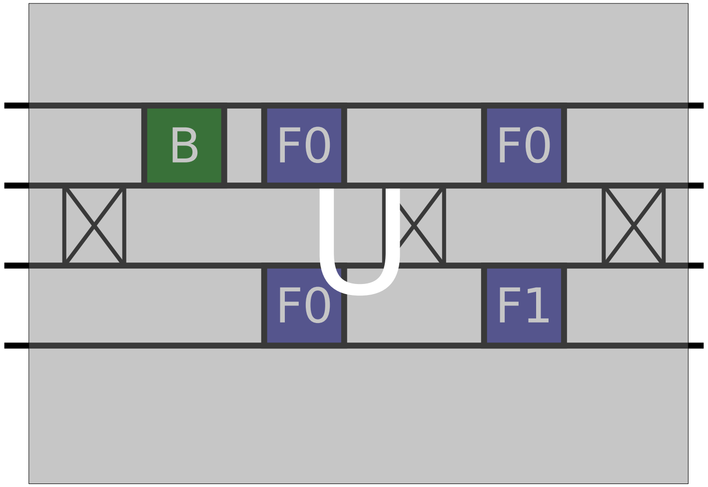
A quantum computer

A quantum computer is a computer that works with qubits instead of bits.

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle$$

By means of quantum gates it can perform a unitary operation on its register of qubits.

$$|\Psi\rangle = \sum_{i_1 i_2 \cdots i_n} c_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle$$



- Single and two qubit gates.
- Local gates.
- Polynomial number of gates.

Quantum and classical computations

Quantum computer

>>

exponentialy faster

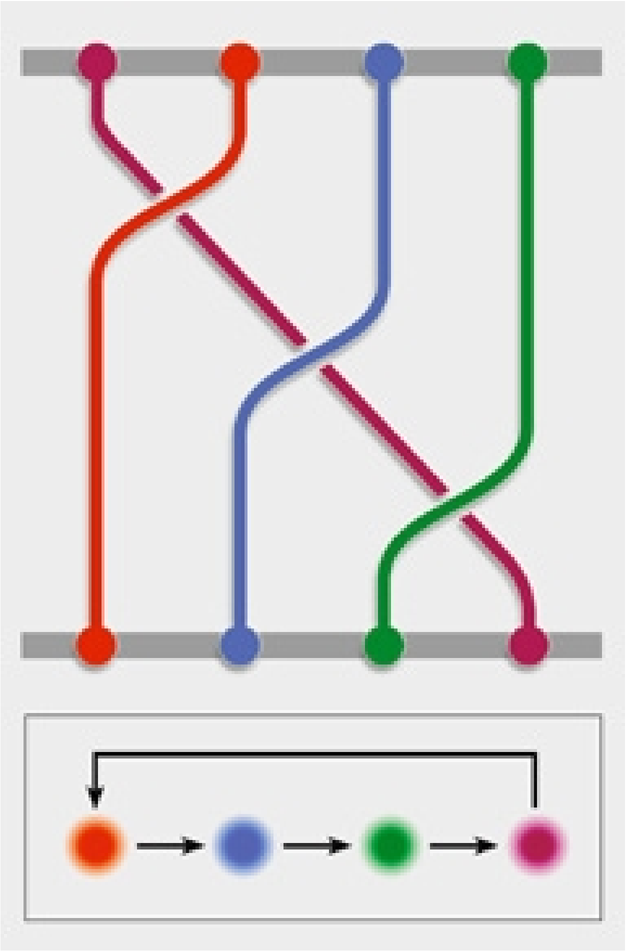
Classical computer

A quantum computer allows us to simulatate quantum mechanics, to solve NP classical problems: factorization (Shor's algorithm), searches in data bases (Grover's algorithm), etc.

Problem: experimentally is very difficult to implement unitary transformations (precision problems, decoherence, etc.).

Idea: to use anyons

Anyons are robust against local perturbations, thus, they do not have the problems of the standard quantum computation paradigm.



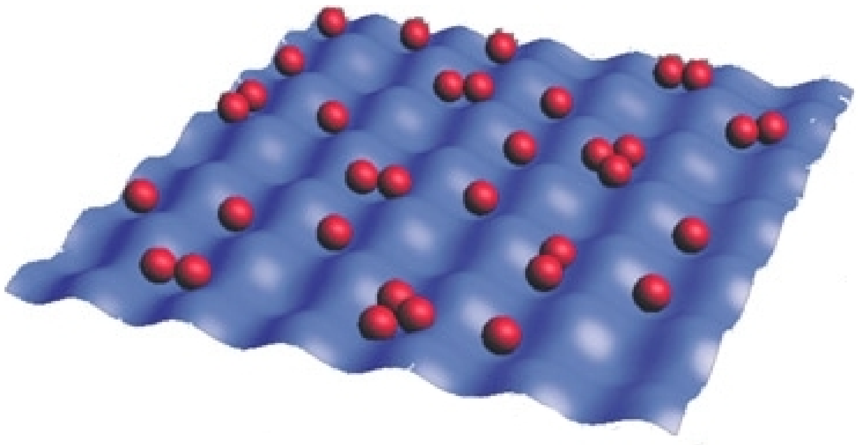
A quantum computation can be realized by braiding anyons.

Creating anyons

Anyons are collective excitations of the FQHE. Nevertheless, electrons are too small to be manipulated experimentally.

The solution to this problem is to realize another quantum system equivalent to the FQHE but accessible experimentally.

A suggestion is to use optical lattices, where the atoms would play the role of electrons and excitations could be created by means of an external laser.



Conclusions

1) The basic concepts related to the Landau symmetry breaking theory have been presented: phase transitions, order, symmetry, order parameter, gapless excitations, etc.

2) We have seen an explicit example of a new kind of order beyond the Landau's paradigm: the FQHE. This new kind of order is called topological order.

3) One of the signs of the topological order is the existence of the anyonic quasiparticles. They are collective excitations that are neither bosons nor fermions.

4) We have realized that anyons would be very useful in order to perform quantum computation. This is new paradigm of computation is called Topological Quantum computation.